Lance Lampert

Winner of the Alice Griswold Award at the 2019 Long Island Mathematics Fair held at Hofstra.

Lance received the highest award given to any Nassau research student who participated in the annual Long Island Mathematics Fair. Lance is a graduate of both our Institute of Creative Problem Solving for Gifted and Talented Students and our Institute of MERIT Research program. His paper was written on a topic familiar to many high school students, the parabola. In general, a parabola is defined by the locus of points equidistant from a given point (focus) and a given line (directrix) not containing the point. Lance asked himself the following "What if" question: What if the point was expanded into a circle (after all a point is the degenerate case of a circle)? Is the result still a parabola? If this is of interest to you, continue to read a synopsis of his paper.

PROBLEM STATEMENT

- Intro / Familiar review of related material
  - What is a loci?
  - What is a parabola? General terms.
    - What isn't a parabola? Common misconceptions
      - Abs(x^3), cosh(x)
  - Derive parabola with vertex at y=0 first
  - Derive parabola with vertex at any y-value
  - All parabolas are self-similar
  - Introduce expanding the focus
- New terms needed
  - Define distance from a point to a polygon or line
    - The distance is the length of the shortest line from the point to the polygon
      - Example? Line?
- "Hook" - what you're investigating
  - Loci of a line and polygons, try to find a pattern
- Bullet list of questions
  - What is the graph of the loci of points equidistant to a square and a line?
  - A pentagon and a line? Is there a pattern?
  - What about rotated polygons?
  - When are they self-similar?
- Transition sentence to end Problem Statement section

A locus (plural: loci) is a figure, graph, or curve that satisfies a predefined mathematical condition or conditions. Many constructions and graphs can be defined as a locus. A line, for example, is defined as the set of all points P satisfying the following condition:

- The distance between *P* and a given point *A* equals the distance between between *P* and a second given point, *B*.

Or

- *P* is *equidistant* from two given points *A* and *B*

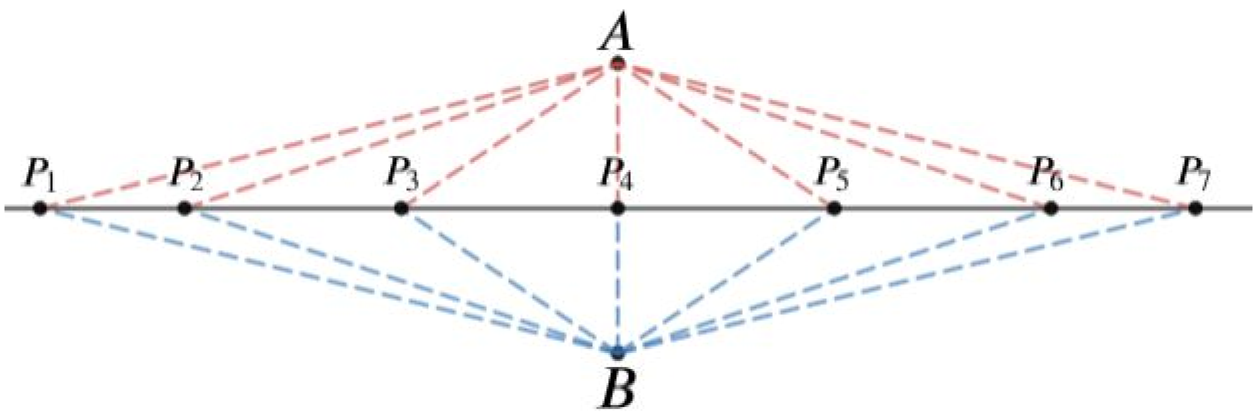

Fig 1. A line as defined as the set of all points equidistant from two arbitrary points, A and B.

In the above figure, the length of each dashed red line (segment $\overline{AP_1}$, for example) is equal to the length of the corresponding blue dashed line (segment $\overline{BP_1}$). This statement is true for all points on the black line, making all points on the line *equidistant* from A and B. This locus can also be seen to be the perpendicular bisector of the segment $\overline{AB}$.

A parabola is another example of a locus. Before looking at the exact geometric definition of a parabola, let's first examine what a parabola *isn't.* It is a misconception that anything that has a general "U" shape is a parabola. Let's look at two examples of things that are very much U-shaped but not at all parabolas.

Let's first look at a real-life example. What is the shape of a cable hanging between two poles of equal height?

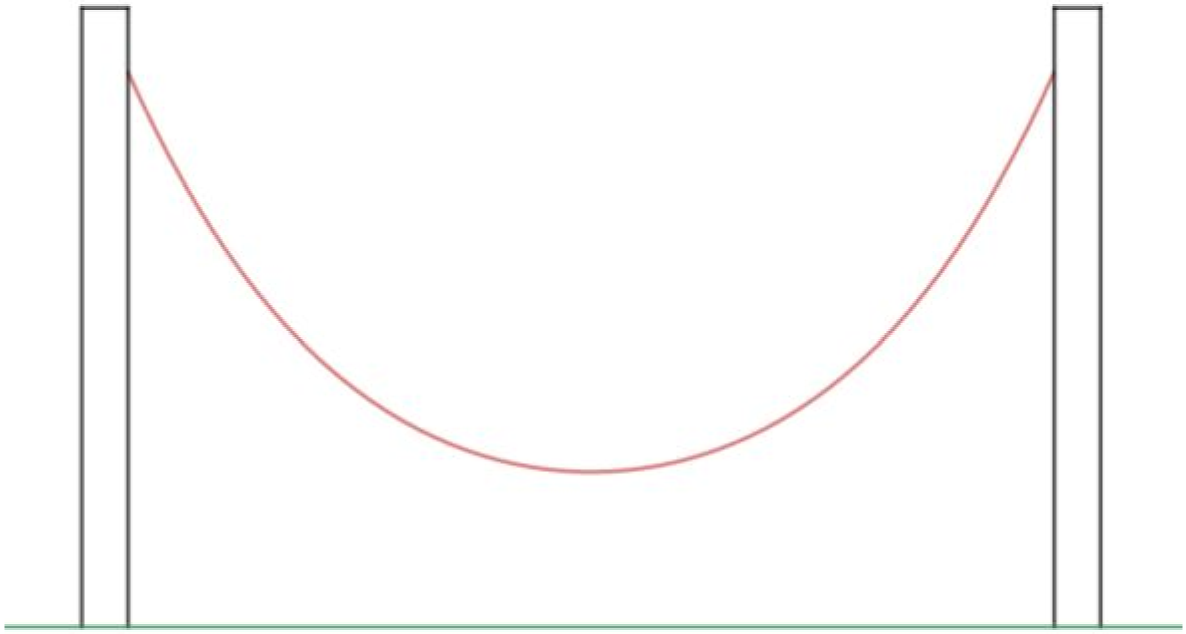

Fig 2. What is the equation describing the cable, the red curve?

You may have looked at a hanging cable and assumed that it was a parabola; I know I have. In actuality, although it has the distinctive U-shape of a parabola, it is actually an equation of a slightly different curve, a *catenary*. More specifically, it is the graph of $y = cosh(x)$, a hyperbolic trigonometric function. What exactly that means isn't important; only the fact that it *isn't* a parabola is.

In fact, assuming that the cable is a parabola isn't a terrible assumption at all. $y = cosh(x)$ can be approximated by a parabola very well — the two functions have very similar shapes. *Fig 2.* with a parabola superimposed upon it (in blue) is shown below.

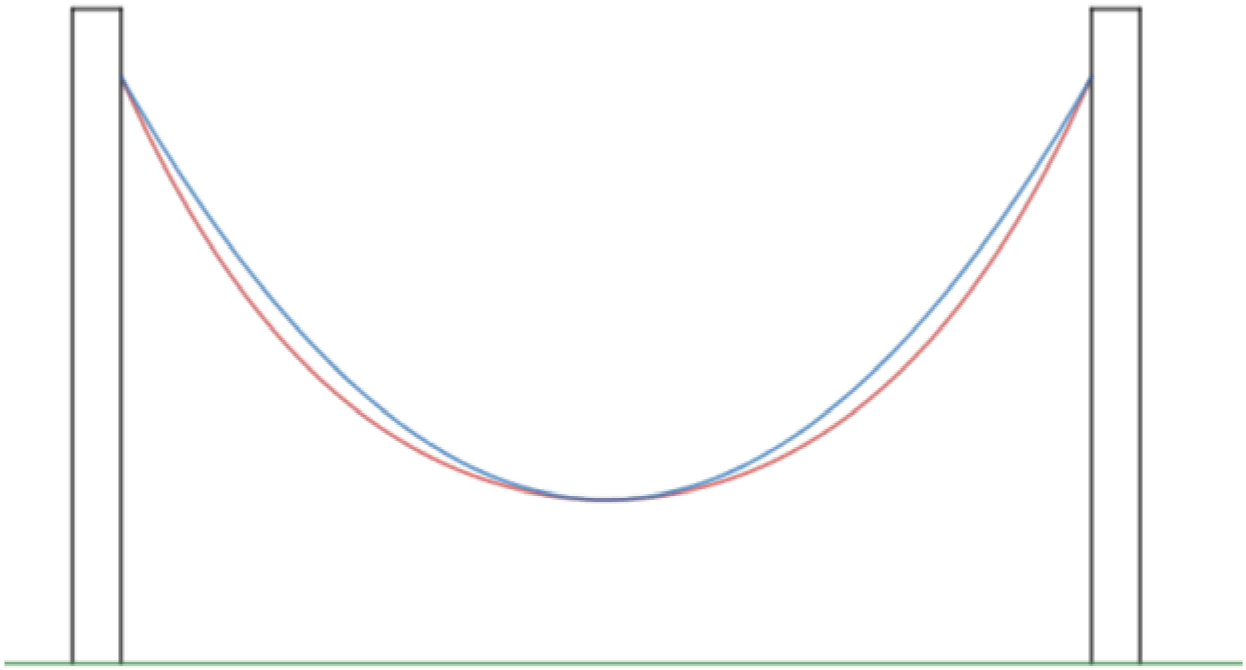

Fig 3. The catenary (in red) and the parabola (in blue) have very similar shapes, but they are different curves.

Another parabola-like curve is $y = |x^3|$. Again, it has a distinctive U-shape that is usually associated with a parabola, but they are not the same function. Algebraically, a parabola always has a degree of 2, while $y = |x^3|$ has a degree of 3. Therefore, the two graphs have to be different.

The discussion of what a parabola *isn't* sets the groundwork for the definition of what a parabola *is.* As mentioned earlier, a parabola is a type of locus. Similarly to the definition of a line, it is defined as the set of points equidistant from two **things** in the plane. However, there is one large difference that gives the parabola a U-shape as opposed to a straight shape. The definition is of a parabola is as follows:

- A parabola is the set of all points P equidistant from a given point F (known as the focus) and a given line ℓ (known as the directrix).

Typically, the directrix is a horizontal line on the x-y plane, although you could rotate the directrix, creating a rotated parabola. We tend to avoid rotated parabolas because they are no longer functions; one value of x can output multiple y-values in a rotated parabola.

This definition should seem very familiar. However, there is one problem yet to be addressed — what does it mean to be a certain distance away from a line? A point can be as far away from a line as you want, depending on how you move the point to the line. Just look at *Fig 1.*: the distance from the point *A* to line *P* changes depending on where on *P* you connect point *A* to. To avoid any confusion, and to maintain consistency, we define the distance from a point to a line as follows:

- The distance from a point P to a line ℓ is the length of the *shortest* line between P and ℓ

This makes the definition of the distance from a point to a line unique, in the sense that, for any point and line, there is only *one* distance between them, because there is only *one* line between the line and point with the shortest length. This line between the point and line is actually a perpendicular to the line. Here's why:

Say we have a point *P* and line ℓ. Draw a perpendicular from ℓ unto *P*. Call the intersection of this perpendicular line and line ℓ point *K*. To prove that $\overline{PK}$ is the shortest line from *P* to ℓ, we assume that there exists a second line, call it $\overline{PJ}$, that intersects line ℓ at point *J* that is not point *K* and prove that this line *must* be longer than $\overline{PK}$. Say *J* is a distance D away from *K* on line ℓ. Since $\overline{PK}$ is perpendicular to ℓ, we can use Pythagorean theorem to find the length of $\overline{PJ}$, $\left|\overline{PJ}\right|$:

$$\left|\overline{PJ}\right|^2 = D^2 + \left|\overline{PK}\right|^2$$

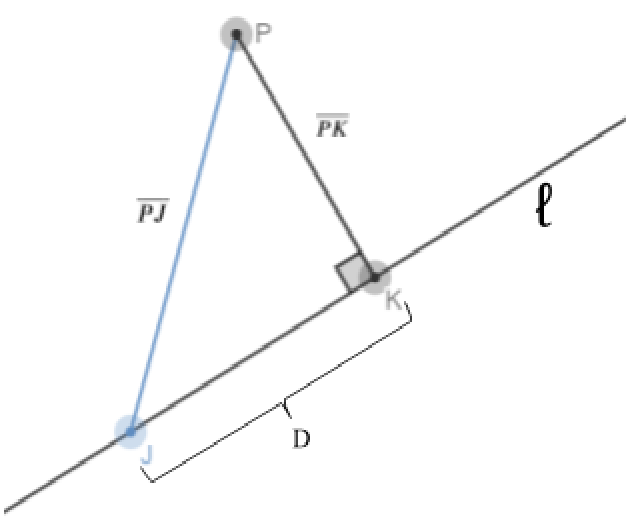

Fig 4. In this diagram, $\left|\overline{PJ}\right|^2 = D^2 + \left|\overline{PK}\right|^2$, where $\overline{PK}$ is the perpendicular from ℓ to *P*.

Note that D and $\left|\overline{PK}\right|$ are positive values, so:

$$D^2 + \left|\overline{PK}\right|^2 > \left|\overline{PK}\right|^2$$

Substitute the LHS for $\left|\overline{PJ}\right|$:

$$\left|\overline{PJ}\right|^2 > \left|\overline{PK}\right|^2$$

Since $\left|\overline{PJ}\right|$ and $\left|\overline{PK}\right|$ are positive, we take the square root of both sides to obtain:

$$\left|\overline{PJ}\right| > \left|\overline{PK}\right|$$

We have proven that the shortest line between a line and a point is the length of the perpendicular line from the line to the point. Therefore, the distance between a line and a point is the length of the perpendicular line from the line to the point.

With this information, we can start deriving the equation for a parabola. For the sake of simplicity, we will start with a parabola with a focus (point *F*) at $(0, p)$ and a directrix (line ℓ) at $y = -p$, where $p > 0$. Recalling back to the definition of a parabola, we set out to find the set of all points (x, y) such that (x, y) is equidistant from $(0, p)$ and $y = -p$. We call the distance from (x, y) to $(0, p)$ D_1 and the distance from (x, y) to $y = -p$ D_2.

To find an equation for D_1, we simply use the distance formula for two points, (x_1, y_1) and (x_2, y_2):

$$d = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$$

Referring to *Fig 5.*, in this case $(x_1, y_1) = (0, p)$, and $(x_2, y_2) = (x, y)$. Substituting:

$$D_1 = \sqrt{(y - p)^2 + (x - 0)^2}$$

Now, the distance from (x, y) to $y = -p$ is, as we proved before, the length of the perpendicular from $y = -p$ to (x, y). This is simply the length of a vertical line from $y = -p$ to (x, y), or the difference in y-coordinates between the line and (x, y), as seen on . Thus, the equation for D_2 is: $D_2 = y + p$

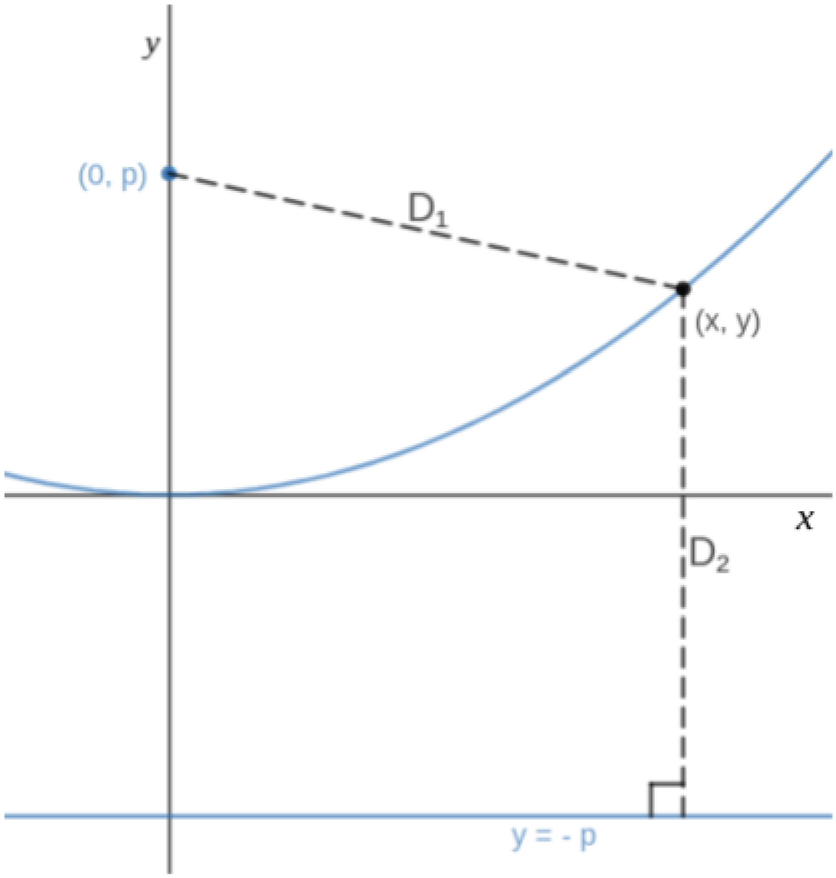

Fig 5. The construction of a parabola with (0, p) as the focus and y = -p as the directrix.

Remember that a parabola is the set of all $(x,\ y)$ such that $(x,\ y)$ is equidistant from the focus and the directrix. Thus, to find a parabola, we set D_1 and D_2 equal to each other:

$$\boldsymbol{D_1 = D_2}$$

Substituting for D_1 and D_2:

$$\sqrt{(y-p)^2 + x^2} = y + p$$

Squaring both sides:

$$(y-p)^2 + x^2 = (y + p)^2$$

Expanding out the terms in red:

$$y^2 - 2py + p^2 + x^2 = y^2 + 2py + p^2$$

Adding like terms:

$$x^2 = 4py$$

This is the standard form of an ellipse with a focus at $(0,\ p)$ and directrix at $y = -p$. This parabola has a vertex at $(0,\ 0)$, the midpoint between $(0,\ p)$, the focus and $(0,\ -p)$, the point on the directrix that shares its x-coordinate with the focus. This parabola would open upward, as indicated by *Fig 5.* If we want the parabola to open downwards, we simply forego the restriction on p, that $p > 0$, and allow p to be negative. This means that, if we want the parabola to open downwards, we put the directrix *above* the focus. This will be important later.

Additionally, if we want to shift the parabola in the plane, we can apply the Graph-Translation Theorem, which states that if we want to shift a graph $y = f(x)$ h units to the right and k units upward, $y = f(x)$ becomes $y - k = f(x - h)$. Note that if h is negative we are shifting the graph to left, and if k is negative we are shifting the graph downwards. Applying this theorem to the equation we found for the parabola:

$$(x-h)^2 = 4p(y-k)$$

This is the general form for an upward- (or downward- if you allow $p < 0$) facing parabola.

Another interesting properties of parabolas is that all parabolas are similar. In precise terms, two figures are similar if one figure can be mapped unto the other using any series of uniform dilations, translations, rotations or reflections.

The similarity of parabolas can be proved relatively easily under an algebraic framework. Assume we have two arbitrary parabolas, one of the form $(x-h)^2 = a(y-k)$ and the other of the form $(x-i)^2 = (y-j)$. To prove that all parabolas are similar, we have to prove that there is a series of transformations that will map these two parabolas onto each other. Two parabolas are mapped onto each other if and only if they have the same equation, thus we must prove that, through a series of transformations, the two parabolas have the same equation.

We translate these two parabolas such that their vertices lie on the origin, resulting in $x^2 = ay$ and $x^2 = by$. Then, we perform a dilation on the first parabola centered at the origin with a scaling factor of $\frac{b}{a}$. To scale a function from the origin with a scaling factor of k, we replace x with $\frac{x}{k}$ and y with $\frac{y}{k}$. Performing the dilation on $x^2 = ay$, we obtain:

$$\left(\frac{x}{\frac{b}{a}}\right)^2 = a\left(\frac{y}{\frac{b}{a}}\right)$$

Multiplying by $\frac{a}{a}$ inside the blue fractions:

$$\left(\frac{ax}{b}\right)^2 = a\left(\frac{ay}{b}\right)$$

Squaring the left side and multiplying like terms:

$$\frac{a^2x^2}{b^2} = \frac{a^2y}{b}$$

Multiplying by $\frac{b^2}{a^2}$ on both sides:

$$x^2 = by$$

Through a series of transformations, we have mapped the first arbitrary parabola unto the second arbitrary parabola. Therefore all parabolas are similar. This is an interesting property not shared by any other conic sections excluding the circle.

Now, we know that a parabola consists of a directrix, which is a line, and focus, which is a point. This is what every high schooler has been taught in every classroom in the country, and it suffices perfectly. But we can take this notion one step further, by considering this construction (between a point and a line) one ideal case out of a class of many possibilities (a degenerate case) - and no, I'm not talking about other conic sections.

Consider the focus of the parabola a circle with radius 0. Therefore, we can expand this focus into a circle by increasing the radius (as we see in Fig. 6), and using a similar definition of a parabola, create an entirely new locus. Thus, our new definition for our locus becomes:

- The set of all point equidistant from a given circle P and a given directrix ℓ

We will call this new locus a circle-focused parabola. We aim to find the equation and properties of this parabola.

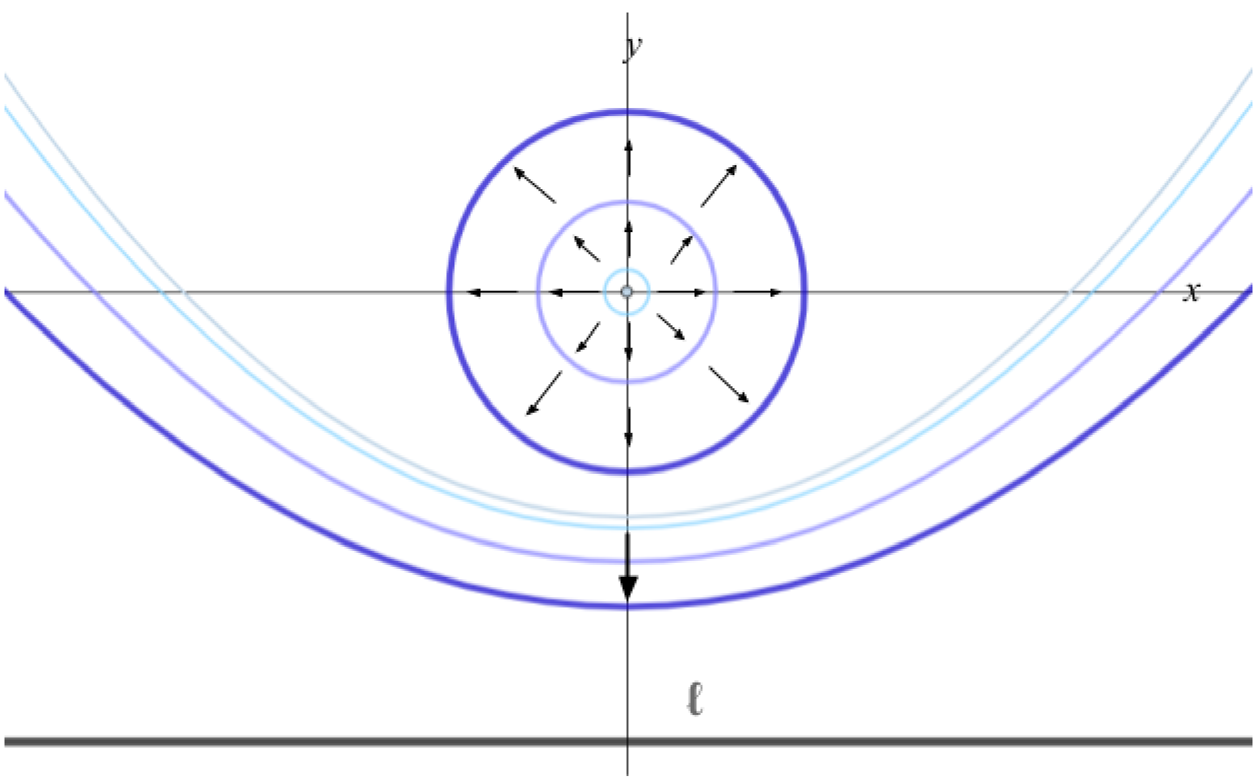

Fig 6. The expansion of a focus of a parabola into a circle, with line ℓ as the directrix.

Furthermore, we can consider this circle a degenerate case of all possible regular polygon (a polygon which is equilateral and equiangular) whose vertices lie on the a circle whose center is the origin, as seen in Fig. 7. Then, similarly, we can define such a polygon-focused parabola as all points equidistant from a line ℓ and a polygon P.

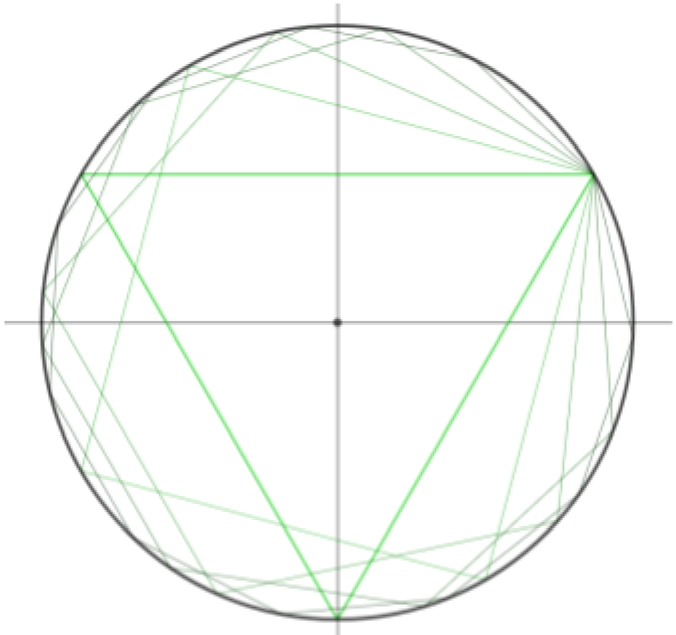

Fig 7. If we consider a circle a polygon with infinite sides, we can construct polygon-focused parabolas.

These polygon-focused parabolas may have different properties when rotated at certain angles around their centers.

The goal of this paper is thus to:

- Find the shape of the circle-focused parabola
- Find the shape of polygon-focused parabolas
- Investigate the properties of these parabolas, such as similarity

RELATED RESEARCH

- Introduce how to draw a parabola with graph paper
    - Extend idea to circles, maybe ellipses?
    - https://pdfs.cpm.org/stuRes/GC/chapter_12/GC_12.1.2.pdf
    - http://www.am.ub.edu/~robert/Documents/ellipse.pdf
    - Is an ellipse that is expanded by 1 on all points compared to another ellipse simply the equation of the ellipse with the coefficients increased by 1?
- Explain, in better depth, the research of the paper I've found
    - Article's background
        - Extensions within — explain the paper in greater detail
        - Extensions beyond
- Some potential problems I'll have to deal with:

        - Define distance between a point and some graph, polygon, etc
        - Have to ignore lines protruding into the polygon to avoid complications
    - Things I want to do:
        - Look at parabolas with polygons as the focus
            - Face of polygon parallel to directrix
                - Polygon has a line of symmetry to a perpendicular line to the directrix
            - Ellipses as focus?
                - Distance is more complicated
                - Perpendicular to an ellipse bisects the angle between the two foci.

We will begin by finding the equation for a circle-focused parabola. First, we must define the "distance" between a point and a 2d figure in the x-y plane. This definition will be essentially identical to our definition of the distance between a point and a line:

- The distance from a point P to a 2D figure D is the length of the *shortest* line between P and D

This definition boils down to a very nice expression for the distance between a point and a circle. First, we will show that the line of shortest length between a point and a circle lies on the infinite line between the center of the circle and the point. In more concrete terms, for any point P and circle ω with center O, let $\overline{QP}$ be the line segment of shortest length between P and ω, where point Q lies on ω. Then O, Q, and P are collinear. This is true if P lies inside or outside ω (the converse of this statement is also true: if O, Q, and P are collinear, then $\overline{QP}$ is the line segment of shortest length between P and ω). A visual interpretation of this argument is shown in Fig 8. This argument should be intuitively obvious but it's important to prove it more formally.

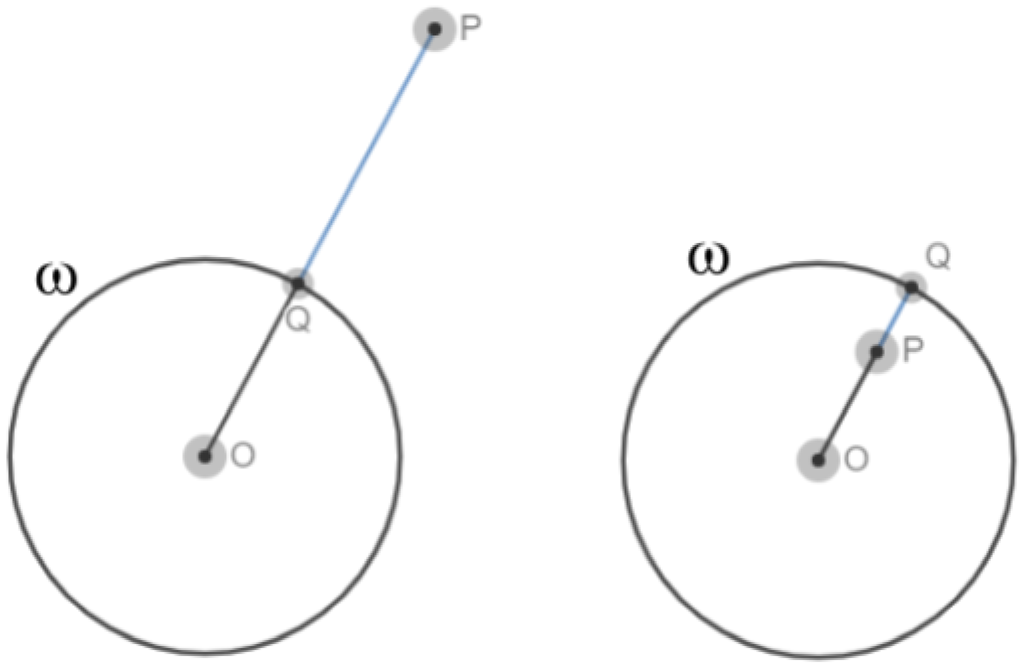

Fig 8. If the length of $\overline{PQ}$ is the distance from *P* to *ω* then *O, P,* and *Q* are collinear, in both cases.

We set out to prove that the line of shortest length between a point (outside the circle), *P*, and a circle with center *O* (which, remember, is the line that defines the "distance" between the point and the circle) intersects the circle at a point *Q*, which is collinear with *O* and *P*. The proof of both cases is very similar. We will use the triangle inequality to prove this, which states that, for any triangle with side lengths x, y, and z, $z < x + y$ (if this weren't true then x and y wouldn't be long enough to connect and form a triangle). Both proofs will also be proofs by contradiction, which I have already used.

Let's first prove the case of the left circle in Fig 8, where we have a point *P* outside the circle. Let's assume there exists a second point, *Z,* on circle ω that isn't point *Q* and isn't collinear with *P* and *O* (refer to Fig 9). We will prove that $|PQ| < |PZ|$ for all *Z*, and thereby $\overline{PQ}$ is the line of shortest length from *P* to ω, so $|PQ|$ must be the distance from *P* to ω. Remember that we define point *Q* to be collinear with *O* and *P*.

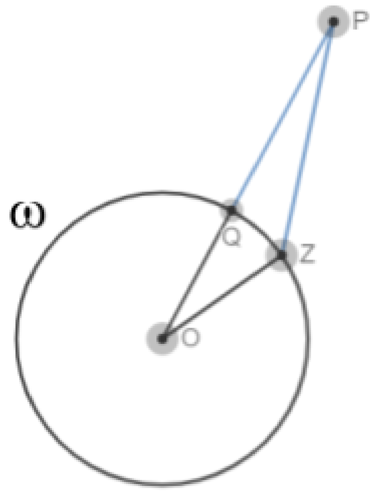

Fig 9. Proof by contradiction for distance from a point to a circle for point outside the circle

We will evoke the triangle inequality on $\triangle OPZ$. Thus:

$$|\overline{OP}| < |\overline{PZ}| + |\overline{OZ}|$$

Since *O, Q,* and *P* are collinear, $|\overline{OP}| = |\overline{OQ}| + |\overline{QP}|$. Substituting the LHS of the inequality:

$$|\overline{OQ}| + |\overline{QP}| < |\overline{PZ}| + |\overline{OZ}|$$

But since $\overline{OQ}$ and $\overline{OZ}$ are radii of circle ω, $|\overline{OQ}| = |\overline{OZ}|$. Substituting $|\overline{OQ}|$ in the LHS of the inequality:

$$|\overline{OZ}| + |\overline{QP}| < |\overline{PZ}| + |\overline{OZ}|$$

Subtracting $\overline{OZ}$ on both sides:

$$|\overline{QP}| < |\overline{PZ}|$$

Therefore, no such *Z* on ω exists such that $|\overline{PZ}|$ is the line of shortest length between the point *P* and ω. $|\overline{PQ}|$ is thus the line of shortest length between *P* and ω, so the line of shortest length between a circle and a point outside the circle lies on the line from the point to the center of the circle.

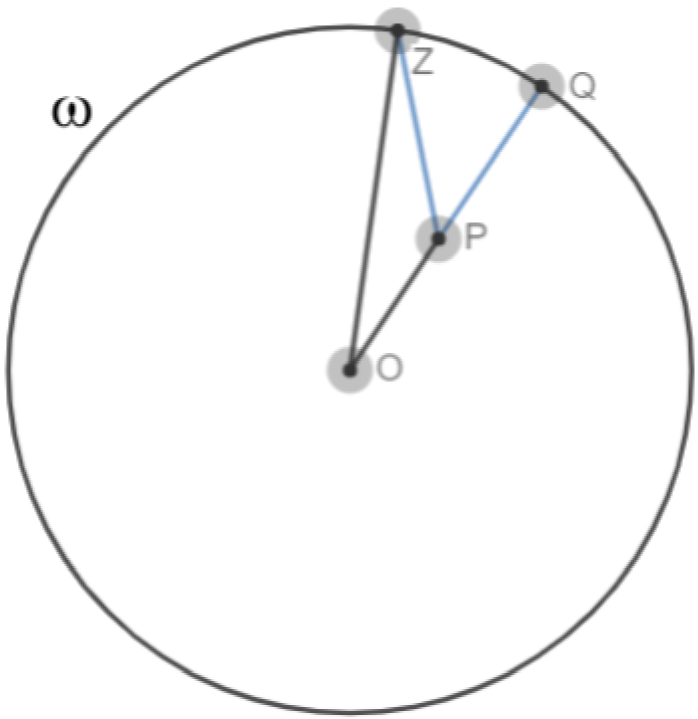

Fig 10. Proof by contradiction for distance from a point to a circle for point inside the circle.

We go about proving that the line of shortest length between a circle and a point inside the circle lies on the line from the center of the circle to the point in a very similar way. Referring to Fig 10, with a circle ω with center *O* and point *P* inside the circle, there exists a point *Q* on ω such that *O, P,* and *Q* are collinear. We assume there exists a second point, *Z*, on ω that isn't collinear with *O* and *P*. We will prove that for all *Z*, $|\overline{PQ}| < |\overline{PZ}|$, so $\overline{PQ}$ is the line of shortest length from *P* to ω , and $|\overline{PQ}|$ is the distance from P to ω .

We evoke the triangle inequality on $\triangle OPZ$. Thus:

$$|\overline{OZ}| < |\overline{OP}| + |\overline{PZ}|$$

Since $\overline{OZ}$ and $\overline{OQ}$ are both radii of ω , $|\overline{OZ}| = |\overline{OQ}|$. Making this substitution on the LHS of the inequality:

$$|\overline{OQ}| < |\overline{OP}| + |\overline{PZ}|$$

Because *O, P,* and *Q* are collinear, $|\overline{OQ}| = |\overline{OP}| + |\overline{PQ}|$. Making this substitution on the LHS of the inequality:

$$|\overline{OP}| + |\overline{PQ}| < |\overline{OP}| + |\overline{PZ}|$$

Subtracting $|\overline{OP}|$ from both sides of the equation:

$$\left|\overline{PQ}\right| < \left|\overline{PZ}\right|$$

Therefore, $\overline{PQ}$ is the line of shortest length from *P* to ω. We have now proven that the line of shortest length between a point and a circle lies on the line from the center of the circle to the point when the point is both inside and outside the circle.

Note that there are actually two points that are collinear to *O* and *P* on ω on both cases. Other than *Q*, we can rotate *Q* 180 degrees on *O* around the circle. This point is also collinear with the circle. However, this point is clearly further away from *P* than *Q* is in both cases (with the exception that *P* lies on *O*, in which case every point on ω is the same distance away from *P)*, so we ignore it and only consider *Q*.

This property of distance from a point to a circle gives us a simple expression for the distance from a point to a circle. Referring back to Fig 8, we first consider the case on the left. If we have a circle ω with radius r and center *O*, for a point *P* outside the circle, the line of shortest distance from *P* to ω intersects ω at *Q*. Since *O, Q,* and *P* are collinear:

$$\left|\overline{OP}\right| = \left|\overline{OQ}\right| + \left|\overline{QP}\right|$$

We call $\left|\overline{QP}\right|$ D_{out} to indicate that it is the distance from point *P* outside circle ω to the circle. Also, $\left|\overline{OQ}\right| = r$, because $\overline{OQ}$ is a radius of ω. Substituting and isolating D_{out}:

$$D_{out} = \left|\overline{OP}\right| - r$$

Now, to take care of when *P* is in the inside of the circle (right circle in Fig 8), we note that (since, again *O, P,* and *Q* are collinear) that:

$$\left|\overline{OQ}\right| = \left|\overline{OP}\right| + \left|\overline{PQ}\right|$$

Again, we call $\left|\overline{PQ}\right|$ D_{in}, and note that $\left|\overline{OQ}\right| = r$. Substituting and isolating D_{in},:

$$D_{in} = r - \left|\overline{OP}\right|$$

These two expressions can be combined to give one simple expression for *D* in all cases:

$$D = \left|\ \left|\overline{OP}\right| - r\right|$$

Why can we combine these two with an absolute value? Notice that the expression for D_{in} is the negative of the expression for D_{out}. Also, D_{in} and D_{out} are always positive. However, the domains for D_{in} and D_{out} are different. When $\left|\overline{OP}\right| \geq r$, because point *P* is outside the circle, we use the expression for D_{out}, namely

$$D = \left|\overline{OP}\right| - r$$

Note we include the edgecase $\left|\overline{OP}\right| = r$, when $D = 0$. When $\left|\overline{OP}\right| < r$, because point *P* is inside the circle, we use the formula for D_{in}:

$$D = r - \left|\overline{OP}\right|$$

If we factor out a negative sign from the RHS of this, and rearrange the terms, we obtain:

$$D = -\left(\left|\overline{OP}\right| - r\right)$$

Thus we can represent our formula for *P* as a piecewise function:

$$D = \begin{cases} \left|\overline{OP}\right| - r & \left|\overline{OP}\right| \geq r \\ -\left(\left|\overline{OP}\right| - r\right) & \left|\overline{OP}\right| < r \end{cases}$$

Now if we subtract r from both sides of both conditional statements we obtain:

$$D = \begin{cases} \left|\overline{OP}\right| - r & \left|\overline{OP}\right| - r \geq 0 \\ -\left(\left|\overline{OP}\right| - r\right) & \left|\overline{OP}\right| - r < 0 \end{cases}$$

Note that this is the very definition of $f(x) = |x|$, if we let $x = \left|\overline{OP}\right| - r$. Thus,

$$D = \left|\, \left|\overline{OP}\right| - r \right|$$

SUGGESTIONS FOR FURTHER RESEARCH

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